Show that $S = \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_{2\times2} / x+y=0 \}$ is a sub space of $M_{2\times2}$

$\Rightarrow$

Let,

$\alpha, \beta \in S$ and $a \in \mathbb{R}$ where

$\alpha = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$ $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ $\left[\begin{array}{l} \text{Where } \lambda_1 + \lambda_2 = 0 \text{ ------ (i)} \\ \text{Where } \beta_1 + \beta_2 = 0 \text{ ------ (ii)} \end{array} \right]$

$\lambda + \beta \begin{pmatrix} \lambda_1 + \beta_1 & \lambda_2 + \beta_2 \\ \lambda_3 + \beta_3 & \lambda_4 + \lambda_4 \end{pmatrix} \in S$

Now,

$L.H.S = \lambda_1 + \beta_1 + \lambda_2 + \beta_2$

$= (\lambda_1 + \lambda_2) + (\beta_1 + \beta_2)$

$= 0$ R.H.S [By (i) and (ii)]

$\therefore \lambda + \beta \in S$ ---- (A)

Again $a\lambda$

$= \begin{pmatrix} a\lambda_1 & a\lambda_2 \\ a\lambda_3 & a\lambda_4 \end{pmatrix}$

L.H.S $a\lambda_1 + a\lambda_2$

$= a(\lambda_1 + \lambda_2)$

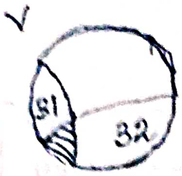
$= a \cdot 0$ [By (i)]

$= 0$ R.H.S

$\therefore a\lambda \in S$ ---- (B)

S is a sub space of $M_{2\times2}$.

2. Show that the union of two sub-sets of a vectors space in is not a vector space in → ..... general though the intersection in a vector space.



$S_1 \cap S_2$ vector Space

$S_1 \cap S_2 \quad \lambda \in S_1$ & S_2

$\beta \in S_1$ & S_2

Since, S_1 is a Sub space

$\lambda + \beta \in S_1$ ---(i) $a\lambda \in S_1 \ \forall a \in R$ ----(iii)

Again, S_2 is a sub space

$\lambda + \beta \in S_2$ ---(ii), $a\lambda \in S_2 \ \forall a \in R$ --- (iv)

Now, (i) & (ii) $\Rightarrow \lambda + \beta \in S_1 \cap S_2$

(iii) & (iv) $\Rightarrow a\lambda \in S_1 \cap S_2$

$S_1 \cap S_2$ is a sub space of V

$S_1 \cup S_2$ is not a Sub space of V always

$V = S_1 = \{(0, y) / y \in V\}$

$S_2 = \{(x, 0) / x \in V\}$

$S_1 \cup S_2 = \{(0, y)\} \cup \{(x, 0)\}$

$\lambda = (0, p)$ where $p \in V$

$\beta = (p, 0)$ where $p \in V$

$\lambda + \beta = (0, p) + (p, 0)$

$= (p, p) \notin S_1$ & S_2

$\lambda + \beta \notin S_1 \cup S_2$

3. What do you mean by linear Combination in a vector space ? Express $(-1,2,4)$ as the line as combination of α, β, γ where $\alpha = (-1,2,0)$, $\beta = (0,-1,1)$ and $\gamma = (3,-4,2) \in \mathbb{R}^3$.

Deffination (Linear combination): Let a vector space V over the field field K. A Vector v is called the linear combination of the vactor $U_1, U_2 --- U_n$ where $v, v_1, v_2 --- U_n \in V$ if for some scalars $a_1, a_2 --- a_n$ in K. We get

$$v = a_1 U_1 + a_2 U_2 + a_3 U_3 + ---- a_n U_n$$

if $v = (3,7,-4)$ the vactor in $\mathbb{R}^3$ then express it as the linear combination of u_1, U_2, U_3 as $\mathbb{R}^3$

Where $U_1 = (1,2,3)$

$U_2 = (2,3,7)$

$U_3 = (3,5,6)$

or,

Let,

$\alpha_1, \alpha_2 --- \alpha_n$ be n number vectors in a vector space v then

$$c_1 . a_1 + c_2 . \alpha_2 + ---- + c_n . \alpha_n = \Sigma c_i . \alpha_i \; (i=1)$$

is called a linear combination of the vectors $\alpha_1, \alpha_2 --- \alpha_n$ where $c_1, c_2 --- c_n$ are n number of reals.

The linear combination can be writh as

$$(-1,2,4) = x(-1,2,0) + y(0,-1,1) + z(3,-4,2)$$

$$\Rightarrow (-1,2,4) = (-x, 2x, 0) + (0,-y,y) + (3z,-4z,2z)$$

$$\Rightarrow (-1,2,4) = (-x+3z, 2x-y-4z, y+2z)$$

$\therefore$ We have
$$-x+3z = -1$$
$$2x-y-4z = 2$$
$$y+2z = 4.$$

$$\Delta = \begin{vmatrix} -1 & 0 & 3 \\ 2 & -1 & -4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= -1(-2-(-4)) - 0(4-(-4\times 0)) + 3(2-0)$$

$$= -2+6$$

$$= 4$$

$$\Delta x = \begin{vmatrix} -1 & 0 & 3 \\ 2 & -1 & -4 \\ 4 & 1 & 2 \end{vmatrix}$$

$$= -1(-2-(-4)) - 0(4-(-4\times 4)) + 3(2-4)$$

$$= 16$$

$$\Delta y = \begin{vmatrix} -1 & -1 & 3 \\ 2 & 2 & -4 \\ 0 & 4 & 2 \end{vmatrix}$$

$$= -1(4+16) + 1(4-0) + 3(8-0)$$

$$= -20+4+24$$

$$= 8$$

$$\Delta z = \begin{vmatrix} -1 & 0 & -1 \\ 2 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix}$$

$$= -1(-4-(2)) - 0 - 1(2-0)$$

$$= 6-2$$

$$= 4.$$

$$x = \frac{\Delta x}{\Delta} = \frac{16}{4} = 4$$

$$y = \frac{\Delta y}{\Delta} = \frac{8}{4} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{4}{4} = 1$$

$\therefore (-1, 2, 4) = 4(-1, 2, 0) + 2(0, -1, 1) + 1(3, -4, 2).$

Define the terms linearly dependent and independent set of vectors. Show that $\{(1,2,2), (2,1,2), (2,2,1)\}$ is a set of linearly independent vectors.

$\Rightarrow \quad (1,2,2), \ (2,1,2), \ (2,2,1)$

$x(1,2,2) + y(2,1,2) + z(2,2,1)$

Hence the system of equations.

$$x + 2y + 2z = 0$$
$$2x + y + 2z = 0$$
$$2x + 2y + z = 0$$

$$\Delta = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$= 1\cdot(1-4) - 2\cdot(2-4) + 2\cdot(4-2)$

$= -3 + 4 + 4$

$= -3 + 8$

$= 5.$

$$\Delta x = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 0 \qquad \Delta y = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \qquad \Delta z = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} = 0$$

$\therefore$ Hence the given set of vector are linearly independent.

Find the value of k such that $\{\lambda_1, \lambda_2, \lambda_3\}$ is linearly independent where

$\lambda_1 = (0, 1, a)$

$\lambda_2 = (1, a, 1)$

$\lambda_3 = (a, 1, 0)$

$$\therefore A = \begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix}$$

$\Rightarrow 0(0-1) - 1(0-a) + a(1-a^2) = 0$

$\Rightarrow 0 + a + a - a^3 = 0$

$\Rightarrow 2a - a^3 = 0$

$\Rightarrow a(2-a^2) = 0$

$\Rightarrow a\{(\sqrt{2})^2 - (a)^2\} = 0$

$\Rightarrow a(\sqrt{2}+a)(\sqrt{2}-a) = 0$

$a = 0$

or, $a + \sqrt{2} = 0$

$\Rightarrow a = -\sqrt{2}$

or, $a - \sqrt{2} = 0$

$\Rightarrow a = \sqrt{2}$

$\therefore$ Hence, the required value of a are $0, -\sqrt{2}, \sqrt{2}$.

6. What is the basis of vectors space? Define the term dimantion in a vector space?

⇒ Basis of vector space:

Let V be a vector space. A collections of vectors $[\alpha_1, \alpha_2 \cdots\cdots \alpha_n]$ is said to form a basis of V if $\alpha_1, \alpha_2 \cdots \alpha_n$ are linearly independent and if they generate V.

Dimension of a vector Space:

The number of vectors present in a basis of a vector space is called the dimension of V. It is denoted by dim (V).

Find the basis and dimension of the sub-space

$$W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + 2y + z = 0,\ 2x + y + 3z = 0\}$$

⇒ Let $\alpha \in S$ then

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 + 2\alpha_2 + \alpha_3 = 0$

$2\alpha_1 + \alpha_2 + 3\alpha_3 = 0$

$$\frac{\alpha_1}{6-1} = \frac{\alpha_2}{2-3} = \frac{\alpha_3}{1-4} = k$$

$$\Rightarrow \frac{\alpha_1}{5} = \frac{\alpha_2}{-1} = \frac{\alpha_3}{-3} = k$$

$\therefore \alpha = (5k, -k, -3k)$

$\Rightarrow \alpha = k(5, -1, 3)$

$= k\beta$

where, $\beta = (5, -1, -3)$

We, now ~~...~~ show that $\{\beta\}$ is a basis of S

$a(5, -1, 3) = 0 = (0, 0, 0)$

$\therefore a = 0$

Hence, $\{\beta\}$ is linearly Independent.

Now, to show that $\{\beta\}$ generates S, we show $L\{\beta\} = S$

$\lambda \in S$, and $\lambda = k\beta \Rightarrow \lambda \in L(\beta)$

$\therefore S \subseteq L(\beta)$

$\beta \in 2(\beta)$ Then.

$L.H.S :- 2.5 + (-1) + 3(-3)$

$= 10 - 1 - 9$

$= 0$ (R.H.S)

$\beta \in S \therefore L(\beta) \subseteq S$

Hence, $L(\beta) = S$

Hence, $\{\beta\}$ is a basis of S

Hence, $\dim(S) = 1$.

1. (i) one real root m_1 — $c_1 e^{m_1 x}$

(ii) two real roots $m_1 \neq m_2$ — $c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(iii) three real roots m_1, m_2, m_3 — $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

2. (i) two real equal roots — $(c_1 + x c_2) e^{mx}$

(ii) three equal real roots $m_1 = m_2 = m_3 = m$ — $(c_1 + x c_2 + x^2 c_3) e^{mx}$

3. (i) pair of complex roots $\alpha \pm i\beta$ — $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$$= e^{-\frac{m}{\sqrt{2}}x}\left[c_1 \cos\left(\frac{m}{\sqrt{2}}x\right) + c_2 \sin\left(\frac{m}{\sqrt{2}}x\right)\right] + e^{\frac{m}{\sqrt{2}}x}\left[c_3 \cos\left(\frac{m}{\sqrt{2}}x\right) + c_4 \sin\left(\frac{m}{\sqrt{2}}x\right)\right]$$

$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

or $c_1 e^{\alpha x} \cos(\beta x + c_2)$

or $c_1 e^{\alpha x} \sin(\beta x + c_2)$

(ii) two pairs of equal complex roots $\alpha \pm i\beta,\ \alpha \pm i\beta$ — $e^{\alpha x}\left[(c_1 \cos\beta x + c_2 \sin\beta x) + (x c_3 \cos\beta x + x c_4 \sin\beta x)\right]$

$$= e^{\alpha x}\left[(c_1 + x c_3)\cos\beta x + (c_2 + x c_4)\sin\beta x\right]$$

(i) $\alpha \pm \sqrt{\beta}$ — $e^{\alpha x}\left[c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta}\right]$

(ii) $\alpha \pm \sqrt{\beta}$ — $e^{\alpha x}\left[c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta}\right]$

or $c_1 e^{\alpha x} \cosh(x\sqrt{\beta} + c_2)$

or $c_2 e^{\alpha x} \sinh(x\sqrt{\beta} + c_1)$

$$e^{\alpha x}\left[(c_1 + x c_2)\cosh x\sqrt{\beta} + (c_3 + x c_4)\sinh x\sqrt{\beta}\right]$$

$m^3 + 6m^2 + 11m + 6 = 0$

$\Rightarrow (m+1)(m+2)(m+3) = 0$

$\therefore m = -1, -2, -3$

Hence the general soln is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

$m^3 + 6m^2 + 11m + 6$

$= m^3 + m^2 + 5m^2 + 5m + 6m + 6$

$= m^2(m+1) + 5m(m+1) + 6(m+1)$

$= (m^2 + 5m + 6)(m+1)$

$= (m^2 + 3m + 2m + 6)(m+1)$

$= (m+3)(m+2)(m+1)$

$D^3 + 3D^2 + 3D + 1 = 0$

$\Rightarrow (D+1)^3 = 0 \Rightarrow D = -1, -1, -1.$

$\therefore$ y.s. $y = (c_1 + c_2 x + c_3 x^2) e^{-x}$

$D^4 - D^3 - 9D^2 - 11D - 4 = 0$

$\Rightarrow (D+1)^3 (D-4) = 0 \quad \therefore D = 4, -1, -1, -1.$

Hence the auxiliary equation is $y = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{4x}$

$1 + 1 - 9 + 11 - 4 =$

$= 13 - 13 = 0$

$(D+1)$ D^3

$D^4 - D^3 - 9D^2 - 11D - 4$

$\underline{D^4 \pm D^3}$

$-2D^3 - 9D^2$

$1 + 2 - 3 - 4 + 4$

$D^4 - 5D^2 + 4 = 0$

$\Rightarrow D^4 - 4D^2 - D^2 + 4 = 0$

$\Rightarrow D^2(D^2 - 4) - 1(D^2 - 4) = 0$

$\Rightarrow (D^2 - 4)(D^2 - 1) = 0$

$\Rightarrow (D+2)(D-2)(D+1)(D-1) = 0$

$\therefore D = 2, -2, 1, -1$

$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}$

$D^4 + 2D^3 - 3D^2 - 4D + 4 = 0$

$\Rightarrow D^4 - D^3 + 3D^3 - 3D^2 - 4D + 4 = 0$

$\Rightarrow D^3(D-1) + 3D^2(D-1) - 4(D-1) + 0 \; (D-1) = 0$

$\Rightarrow (D-1)(D^3 + 3D^2 - 4) = 0$

$\Rightarrow (D-1)(D-1)(D^2 + 4D + 4) = 0$

$\Rightarrow (D-1)^2 (D+2)^2 = 0$

$\therefore D = 1, 1, -2, -2$

$y = (c_1 + x c_2) e^{x} + (c_3 + x c_4) e^{-2x}$

$m^3 - 8 = 0$

$\Rightarrow (m-2)(m^2 + 2m + 4) = 0$

$\therefore m = 2, \dfrac{-2 \pm \sqrt{4 - 16}}{2} = 2, \dfrac{-1 \pm i\sqrt{3}}{}$

$\therefore y = c_1 e^{2x} + e^{-x} (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

The auxiliary eqn is

$D^4 + m^4 = 0$

$\Rightarrow (D^2 + m^2)^2 - 2D^2 m^2 = 0$

$\Rightarrow (D^2 + m^2)^2 - (\sqrt{2} Dm)^2 = 0$

$D = \frac{m}{\sqrt{2}} \pm i \frac{m}{\sqrt{2}}$

$\Rightarrow (D^2 + m^2 + \sqrt{2}Dm)(D^2 + m^2 - \sqrt{2}Dm) = 0$

$\therefore D^2 + \sqrt{2}Dm + m^2 = 0$

$\Rightarrow D = \dfrac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}$

$= -\frac{m}{\sqrt{2}} \pm i \frac{m}{\sqrt{2}}$

$D^4 - m^4 = 0$

$\Rightarrow (D+m)(D-m)(D^2+m^2) = 0$

$\therefore D = -m, m, \pm im$

$\therefore y = c_1 e^{-mx} + c_2 e^{mx} + (c_3 \cos mx + c_4 \sin mx)$

$(D^2+1)^2 = 0$

$\Rightarrow (D^2+1)(D^2+1) = 0$

$\therefore D = \pm i, \pm i$

$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)$

8. $(D^2+1)^2 = 0 \Rightarrow D^2+1 = 0$ (twice)

$\Rightarrow D = \pm i$ (twice)

$y = (c_1 + x c_2)\cos x + (c_3 + x c_4)$

8.(b) $(D^2 - 2D + 5)^2 = 0 \Rightarrow D^2 - 2D + 5 = 0$ twice

$\therefore D = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm i4}{2} = 1 \pm i2$

$\therefore y = e^{x}\{(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\}$

9. The auxiliary equation is

$D^4 - 6D^3 + 12D^2 - 8D = 0$

$\Rightarrow D(D^3 - 6D^2 + 12D - 8) = 0$

$\Rightarrow D(D-2)^3 = 0$

$\therefore D = 0, 2, 2, 2$

Hence the required C.F. $y = c_1 + (c_2 + x c_3 + x^2 c_4)e^{2x}$

. $D^6 - 1 = 0 \Rightarrow (D^3-1)(D^3+1) = 0 \Rightarrow (D-1)(D+1)(D^2+D+1)($

$\therefore D-1 = 0$ $\quad D+1=0$ $\quad D^2+D+1=0$ $\quad D^2-D+1=0$

$\Rightarrow D = 1$ $\quad \Rightarrow D = -1$ $\quad \Rightarrow D = \dfrac{-1 \pm i\sqrt{3}}{2}$ $\quad \Rightarrow D = \dfrac{1 \pm i\sqrt{3}}{2}$

Hence the required soln is

$y = c_1 e^{x} + c_2 e^{-x} + e^{-x/2}\left[c_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 \sin(\right.$

$+ e^{x/2}\left[c_5 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_6 \sin(\right.$

(11)(a) $D^4+8D^2+16=0$

$\Rightarrow (D^2+4)^2=0$

$\Rightarrow D=\pm i2$ (twice)

Hence the required soln.

$y=(c_1+xc_2)\cos 2x + (c_3+xc_4)\sin 2x$

(b) $(D^2+D+1)^2=0$

$\Rightarrow D^2+D+1=0$ (twice)

$\therefore D=\frac{-1\pm i\sqrt{3}}{2}$ (twice)

$y=e^{-x}\left[(c_1+xc_2)\cos\left(\frac{\sqrt{3}x}{2}\right)+(c_3+xc_4)\sin\left(\frac{\sqrt{3}x}{2}\right)\right].$

2) $(D^2+1)^3(D^2+D+1)^2=0$

$\therefore (D^2+1)^3=0$

$\Rightarrow D^2+1=0$ (thrice)

$\Rightarrow D=\pm i$ (thrice)

$D^2+D+1=0$ (twice)

$\Rightarrow D=\frac{-1\pm i\sqrt{3}}{2}$ (twice)

$y=[(c_1+xc_2+x^2c_3)\cos x+(c_4+xc_5+x^2c_6)\sin x] + e^{-x/2}\left[(c_7+xc_8)\cos\left(\frac{\sqrt{3}x}{2}\right)+(c_9+xc_{10})\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

The given equation can be written as

$D^2 i+\frac{R}{L}Di+\frac{1}{Lc}i=0$, where $D\equiv\frac{d}{dt}$ & $D^2\equiv\frac{d^2}{dt^2}$.

Hence, the auxiliary equation is

$m^2+\frac{R}{L}m+\frac{1}{Lc}=0$

$\Rightarrow m=\frac{-R/L\pm\sqrt{R^2/L^2-4/(Lc)}}{2}$

$=\frac{1}{2}\left[-R/L\pm\sqrt{\frac{4L}{L^2c}-\frac{4}{(Lc)^2}}\right]$

$=\frac{1}{2}\left[-\frac{R}{L}\pm 2\sqrt{1/Lc-1/(Lc)}\right]$

$=-R/2L$

$\therefore m=-\frac{R}{2L}$ (twice). Hence the required soln. is

$i=(c_1+xc_2)e^{-\frac{R}{2L}t}$

~~$X+2\sqrt{3}-1+\sqrt{3}-\sqrt{2}+\sqrt{3}$~~

9. $D^3+D^2(2\sqrt{3}-1)+D(3-2\sqrt{3})-3=0$

$\Rightarrow D^3-D^2+2\sqrt{3}D^2-2\sqrt{3}D+3D-3=0$

$\Rightarrow D^2(D-1)+2\sqrt{3}D(D-1)+3(D-1)=0$

$\Rightarrow (D-1)\{D^2+2\sqrt{3}D+3\}=0$ $\therefore D=1$

$\Rightarrow (D-1)=0$ or $D^2+2\sqrt{3}D+3=0$

$D=\frac{-2\sqrt{3}\pm\sqrt{12-12}}{2}$

$=-\sqrt{3}$ (twice)

$y=c_1e^x+(c_2+xc_3)e^{-\sqrt{3}x}$

1. $\frac{1}{D-\alpha} X = e^{\alpha x}\int e^{-\alpha x} X \, dx$

(*) $\frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{x^n}{n!} e^{\alpha x}$

(*) $\frac{1}{f(D)} X = \frac{1}{(D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_n)} X = \frac{1}{(D-\alpha_1)\dots(D-\alpha_{n-1})} e^{\alpha_n x}\int e^{-\alpha_n x} X\, dx$

$\frac{1}{f(D)} X = \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \dots + \frac{A_n}{D-\alpha_n}\right) X$

$= A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x} X\, dx + A_2 e^{\alpha_2 x}\int e^{-\alpha_2 x} X\, dx + \dots$

$+ A_n e^{\alpha_n x}\int e^{-\alpha_n x} X\, dx$

1. ~~The given eqn~~

$D^2 + a^2 = 0$

$\Rightarrow D = \pm ia$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$

$D^2 + a^2$
$= D^2 - i^2 a^2$
$= (D+ia)(D-ia)$

P.I. $= \frac{1}{D^2+a^2} \cot ax$

$= \frac{1}{(D+ia)(D-ia)} \cot ax$

$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \cot ax$

$= \frac{1}{2ia}\left[\frac{1}{D-ia}\cot ax - \frac{1}{D+ia}\cot ax\right]$

$\therefore \frac{1}{D-ia}\cot ax = e^{iax}\int e^{-iax}\cot ax\, dx$

$= e^{iax}\int (\cos ax - i\sin ax)\frac{\cos ax}{\sin ax}\, dx$

$= e^{iax}\int\left(\frac{\cos^2 ax}{\sin ax} - i\cos ax\right) dx = e^{iax}\int\left[\frac{1-\sin^2 ax}{\sin ax} - \right.$

$$= e^{iax}\int\left[\operatorname{cosec} ax - \sin ax - i\cos ax\right]dx$$

$$= e^{iax}\left[\frac{1}{a}\log\left\{\tan\left(\frac{ax}{2}\right)\right\} + \frac{1}{a}\cos ax - \frac{i}{a}\sin ax\right]$$

$$= e^{iax}\left[\frac{1}{a}\log\left\{\tan\left(\frac{ax}{2}\right)\right\} + \frac{1}{a}e^{-iax}\right]$$

$$\frac{1}{D-ia}\cot ax = \frac{1}{a}\left[e^{iax}\log\left\{\tan\left(\frac{ax}{2}\right)\right\} + 1\right].$$

$$\frac{1}{D+ia}\cot ax = e^{-iax}\int e^{iax}\cot ax\,dx$$

$$= e^{-iax}\int\left[\cos ax + i\sin ax\right]\frac{\cos ax}{\sin ax}\,dx$$

$$= e^{-iax}\int\left[\frac{1-\sin^2 ax}{\sin ax} + i\cos ax\right]dx$$

$$= e^{-iax}\int\left[\operatorname{cosec} ax - \sin ax + i\cos ax\right]dx$$

$$= \frac{1}{a}e^{-iax}\left[\log\left(\tan\frac{ax}{2}\right) + \cos ax + i\sin ax\right]$$

$$= \frac{1}{a}e^{-iax}\left[\log\left(\tan\frac{ax}{2}\right) + e^{iax}\right]$$

$$= \frac{1}{a}\left[e^{-iax}\log\left(\tan\frac{ax}{2}\right) + 1\right]$$

$$P.I. = \frac{1}{a}\left[e^{iax}\log\left(\tan\frac{ax}{2}\right) - e^{-iax}\log\left(\tan\frac{ax}{2}\right)\right]\frac{1}{2ia}$$

$$= \frac{1}{a}\left[\left(e^{iax} - e^{-iax}\right)\log\left(\tan\frac{ax}{2}\right)\right]\frac{1}{2ia}$$

$$= \frac{2}{a}\left[i\sin ax\log\left(\tan\frac{ax}{2}\right)\right]\frac{1}{2ia}$$

$\cancel{y = C.F. + P.I.}$

$$= \frac{1}{a^2}\sin ax\log\left(\tan\frac{ax}{2}\right).$$

Here, the auxiliary equation is

$$D^2+4=0$$

$$\Rightarrow D = \pm i2$$

Hence the complementary function is given by

$$c_1 \cos 2x + c_2 \sin 2x,$$

where, c_1 & c_2 are arbitrary constants.

Now, the particular integral is given by

$$\frac{1}{D^2+4} \tan 2x$$

$$= \frac{1}{(D-i2)(D+i2)} \tan 2x$$

$$= \frac{1}{4i}\left[\frac{1}{D-i2} - \frac{1}{D+i2}\right] \tan 2x$$

$$= \frac{1}{4i}\left[\frac{1}{D-i2}\tan 2x - \frac{1}{D+i2}\tan 2x\right]$$

Now, $\frac{1}{D-i2} \tan 2x$

$$= e^{i2x}\int e^{-i2x} \tan 2x \, dx$$

$$= e^{i2x}\int [\cos 2x - i\sin 2x] \frac{\sin 2x}{\cos 2x} dx$$

$$= e^{i2x}\int \left[\sin 2x - i\frac{1-\cos^2 2x}{\cos 2x}\right] dx$$

$$= e^{i2x}\int \left[\sin 2x - i\left(\sec 2x - \cos 2x\right)\right] dx$$

$$= e^{i2x} \left[-\frac{\cos 2x}{2} - i\left(\frac{1}{2}\log\tan\left(\frac{\pi}{4}+x\right) - \frac{\sin 2x}{2}\right)\right]$$

$$= -\frac{e^{i2x}}{2}\left[i \log\tan\left(\pi/4 + x\right) + \cos 2x + i \sin 2x \right]$$

$$= -\frac{e^{i2x}}{2}\left[i \log\tan\left(\pi/4 + x\right) + e^{-i2x} \right]$$

$$= -\frac{1}{2}\left[i\, e^{i2x} \log\tan\left(\pi/4 + x\right) + 1 \right]$$

And, $\frac{1}{D+2i} \tan 2x$

$$= -\frac{e^{-i2x}}{2}\left[-i \log\tan\left(\pi/4 + x\right) + e^{i2x} \right]$$

$$= -\frac{1}{2}\left[-ie^{-i2x} \log\tan\left(\pi/4 + x\right) + 1 \right]$$

$$\therefore \text{P.I.} = \frac{-1}{4i \cdot 2}\left[i\left(e^{i2x} + e^{-i2x}\right) \log\tan\left(\pi/4 + x\right) \right]$$

$$= -\frac{1}{8}\left[2\cos 2x \log\tan\left(\pi/4 + x\right) \right]$$

$$= -\frac{1}{4} \cos 2x \log\tan\left(\pi/4 + x\right)$$

$$\therefore y = \text{C.F.} + \text{P.I}$$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log\tan\left(\pi/4 + x\right)$$

$$D^2 - 3D + 2 = 0$$
$$\Rightarrow D^2 - 2D - D + 2 = 0$$
$$\Rightarrow (D-2)(D-1) = 0$$
$$\therefore D = 1, 2$$
$$\therefore C.F. = c_1 e^{x} + c_2 e^{2x}$$
$$P.I. = \frac{1}{D^2 - 3D + 2} \sin e^{-x}$$
$$= \frac{1}{(D-1)(D-2)} \sin e^{-x}$$

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$$= \left[\frac{(D-1)-(D-2)}{(D-1)(D-2)}\right] \sin e^{-x}$$
$$= \left[\frac{1}{D-2} - \frac{1}{D-1}\right] \sin e^{-x}$$

$$\frac{1}{D-2} \sin e^{-x}$$
$$= e^{2x} \int e^{-2x} \sin e^{-x} \, dx \qquad \left[\begin{array}{l} e^{-x} = t \\ \therefore -e^{-x} dx = dt \end{array}\right]$$
$$= -e^{2x} \int t \sin t \, dt$$
$$= -e^{2x} [-t\cos t + \sin t]$$
$$= -e^{2x} [-e^{-x} \cos e^{-x} + \sin e^{-x}]$$

$$\frac{1}{D-1} \sin e^{-x}$$
$$= e^{x} \int e^{-x} \sin e^{-x} \, dx$$
$$= -e^{x} \int \sin t \, dt$$
$$= e^{x} \cos t$$
$$= e^{x} \cos e^{-x}$$

dx

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$$\therefore y = c_1 e^{x} + c_2 e^{2x} + [-e^{2x} (-e^{-x} \cos e^{-x} + \sin e^{-x}) - e^{x} \cos e^{-x}]$$
$$= c_1 e^{x} + c_2 e^{2x} + [\cancel{e^{x} \cos e^{-x}} - e^{2x} \sin e^{-x} - \cancel{e^{x} \cos e^{-x}}]$$
$$= c_1 e^{x} + c_2 e^{2x} - e^{2x} \sin e^{-x}$$

$$\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \quad \text{for } f(a) \neq 0.$$

$f(a) \neq 0$

$$\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}.$$

$f(D) = (D-a)^m$

$f(D) = (D-a)^r \phi(D)$

$\phi(D) \neq 0.$

$$\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

$$\frac{1}{f(D)} e^{ax}, \quad f(a) = 0$$

$$= \frac{1}{(D-a)^r \phi(D)} e^{ax}, \quad \phi(a) \neq 0$$

$$= \frac{1}{(D-a)^r} \frac{e^{ax}}{\phi(a)},$$

$$= \frac{1}{\phi(a)} \frac{1}{(D-a)^r} e^{ax}$$

$$= \frac{1}{\phi(a)} \frac{x^n e^{ax}}{n!}$$

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1. (a) The auxiliary equation is

$$D^2 - 3D + 2 = 0$$

$$\Rightarrow (D-2)(D-1) = 0$$

$\therefore$ $D = 1, 2$, which gives the C.F. $= c_1 e^{x} + c_2 e^{2x}$, where c_1 & c_2 are arbitrary constants.

$$\therefore \text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x}, \quad f(3) = 3^2 - 3^2 + 2 = 2 \neq 0$$

$$= \frac{e^{3x}}{2}$$

Hence the required general sol^n is $y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}$.

(b)

$$4D^2 + 12D + 9 = 0$$

$$\Rightarrow (2D+3)^2 = 0$$

$$\therefore D = -3/2, -3/2$$

$$\therefore \text{C.F.} = (c_1 + x c_2) e^{-\frac{3x}{2}}$$

$$\frac{1}{(2D+3)^2} 144 e^{-3x} = 144 \frac{1}{(2D+3)^2} e^{-3x}$$

$$= 144 \frac{e^{-3x}}{9} = 16 e^{-3x}$$

$$y = (c_1 + x c_2) e^{-\frac{3x}{2}} + 16 e^{-3x}$$

$$D^2 + 2pD + p^2 + q^2 = 0$$

$$\Rightarrow D = \frac{-2p \pm \sqrt{4p^2 - 4p^2 - 4q^2}}{2} = -p \pm iq$$

$$\therefore C.F. = e^{-px}(c_1 \cos qx + c_2 \sin qx)$$

$$P.I. = \frac{1}{D^2 + 2pD + p^2 + q^2} e^{ax} = \frac{e^{ax}}{a^2 + 2pa + p^2 + q^2} = \frac{e^{ax}}{(a+p)^2 + q^2}$$

$$y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + \frac{e^{ax}}{(a+p)^2 + q^2}$$

$$D^2 (D+1)^2 (D^2 + D + 1)^2 = 0$$

$$\therefore D = 0, 0, -1, -1, \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore CF = (c_1 + xc_2) + (c_3 + xc_4)e^{-x} + e^{-x/2}\left[c_5 \cos\frac{\sqrt{3}x}{2} + c_6 \sin\frac{\sqrt{3}x}{2}\right] + e^{-x/2}\left[xc_7 \cos\frac{\sqrt{3}x}{2} + xc_8 \sin\frac{\sqrt{3}x}{2}\right]$$

$$\frac{1}{D^2 (D+1)^2 (D^2 + D + 1)^2} e^x$$

$$\frac{1}{1} \frac{1}{(D+1)^2 (D^2 + D + 1)^2} e^x$$

$$= \frac{1}{4} \frac{1}{(D^2 + D + 1)^2} e^x$$

$$= \frac{1}{4} \cdot \frac{1}{9} e^x$$

$$= \frac{1}{36} e^x.$$

$$(D+2)(D-1)^3 = 0$$

$$\therefore D = 1, 1, 1, -2$$

$$\therefore C.F. = e^x(c_1 + xc_2 + x^2 c_3) + c_4 e^{-2x}$$

$$P.I. = \frac{1}{(D+2)(D-1)^3} e^x$$

$$= \frac{1}{3} \frac{1}{(D-1)^3} e^x$$

$$= \frac{1}{3} \cdot \frac{1}{3!} x^3 e^x$$

$$= \frac{1}{18} x^3 e^x$$

$$\therefore y = (c_1 + xc_2 + x^2 c_3) e^x + c_4 e^{-2x} + \frac{1}{18} x^3 e^x$$

(4)

(3)(b) $(D-1)^2(D^2+1)^2 = 0$

$\therefore D = 1, 1, \pm i, \pm i.$

$C.F. = e^x(c_1 + xc_2) + (c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x$

$$I.P. = \frac{1}{(D-1)^2(D^2+1)^2} e^x = \frac{1}{4}\frac{1}{(D-1)^2}e^x = \frac{1}{4}\frac{x^2}{2!}e^x = \frac{1}{8}x^2e^x$$

$$\therefore y = (c_1 + xc_2)e^x + (c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x + \frac{1}{8}x^2e^x.$$

(4)(a) $(D^2 + D - 2)y = e^x$

The auxiliary equation is

$D^2 + D - 2 = 0$

$\Rightarrow D^2 + 2D - D - 2 = 0$

$\Rightarrow (D+2)(D-1) = 0$

$\therefore D = 1, 2$

Hence the C.F. $= c_1e^x + c_2e^{2x}$,

where c_1 & c_2 are arbitrary constants and C.F. stands for complementary functions.

Now, a particular integral

$$= \frac{1}{D^2 + D - 2}e^x = \frac{1}{(D+2)(D-1)}e^x$$

$$= \frac{1}{3}\frac{1}{D-1}e^x$$

$$= \frac{1}{3}\frac{x}{1!}e^x$$

$$= \frac{x}{3}e^x$$

Hence the required soln. is

$$y = c_1e^x + c_2e^{2x} + \frac{x}{3}e^x.$$

) (a) $D^2 - 3D + 2 = 0$

$\Rightarrow (D-2)(D-1) = 0$

$\therefore D = 1, 2$

$\therefore C.F. = c_1e^x + c_2e^{2x}$

$$\frac{1}{(D-2)(D-1)}(e^x + e^{2x})$$

$$= \frac{1}{(D-2)(D-1)}e^x + \frac{1}{(D-2)(D-1)}e^{2x}$$

$$= -1 \cdot \frac{1}{D-1}e^x + \frac{1}{1}\frac{1}{D-2}e^{2x}$$

$$= -\frac{x}{1!}e^x + \frac{x}{1!}e^{2x}$$

$$= -xe^x + xe^{2x}$$

$$\therefore y = c_1e^x + c_2e^{2x} - xe^x(1 - e^x).$$

(b) $\frac{1}{D^2-3D+2}\cosh x$

$= \frac{1}{(D-1)(D-2)}\left(\frac{e^{x}+e^{-x}}{2}\right)$

$= \frac{1}{2}\left[\frac{1}{(D-1)(D-2)}e^{x} + \frac{1}{(D-1)(D-2)}e^{-x}\right]$

$= \frac{1}{2}\left[-\frac{1}{D-1}e^{x} + \frac{1}{-2}\frac{1}{D-2}e^{-x}\right]$

$= -\frac{1}{2}\left[\frac{1}{D-1}e^{x} + \frac{1}{2}\frac{1}{D-2}e^{-x}\right]$

$= -\frac{1}{2}\left[\frac{xe^{x}}{1} + \frac{1}{2}\cdot\frac{1}{-3}e^{-x}\right]$

$= -\frac{x}{2}e^{x} + \frac{1}{12}e^{-x}$

$\therefore y = c_1e^{x} + c_2e^{2x} - \frac{x}{2}e^{x} + \frac{1}{12}e^{-x}$

) $D^3 - 5D^2 + 7D - 3 = 0$

$\Rightarrow D^3 - D^2 - 4D^2 + 4D + 3D - 3 = 0$

$\Rightarrow D^2(D-1) - 4D(D-1) + 3(D-1) = 0$

$\Rightarrow (D-1)(D^2 - 4D + 3) = 0$

$\Rightarrow (D-1)(D-1)(D-3) = 0$

$\therefore D = 1, 1, 3$

$\therefore C.F. = (c_1 + xc_2)e^{x} + c_4e^{3x}$

$y = (c_1 + xc_2)e^{x} + c_4e^{3x}$

$+ \frac{1}{8}xe^{3x} - \frac{1}{8}x^2e^{x}$

$\frac{1}{(D-1)^2(D-3)}e^{2x}\cosh x$

$= \frac{1}{(D-1)^2(D-3)}e^{2x}\left(\frac{e^{x}+e^{-x}}{2}\right)$

$= \frac{1}{2}\frac{1}{(D-1)^2(D-3)}\left(e^{3x}+e^{x}\right)$

$= \frac{1}{2}\left[\frac{1}{4}\frac{1}{D-3}e^{3x} + \frac{1}{-2}\frac{1}{(D-1)^2}e^{x}\right]$

$= \frac{1}{2}\left[\frac{1}{4}\frac{xe^{3x}}{1!} + \frac{-1}{2}\frac{x^2e^{x}}{2!}\right]$

$= \frac{1}{8}xe^{3x} - \frac{1}{8}x^2e^{x}$

(d) $\dfrac{d^3y}{dx^3} - y = (e^x+1)^2$ can be written as

$(D^3-1)y = (e^x+1)^2$ where $D \equiv \dfrac{d}{dx}$ and so on. Hence the auxilliary equation is

$$D^3-1=0 \Rightarrow (D-1)(D^2+D+1)=0$$

$$\therefore D = 1, \frac{-1\pm i\sqrt{3}}{2}$$

Hence the complementary function $= c_1e^x + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}x}{2} + c_3\sin\frac{\sqrt{3}x}{2}\right]$

where c_1, c_2, c_3 are arbitrary constants.

Now, the particular integral $= \dfrac{1}{D^3-1}(e^x+1)^2 = \dfrac{1}{(D-1)(D^2+D+1)}(e^{2x}+2e^x+1)$

$$= \frac{1}{7}e^{2x} + 2\cdot\frac{1}{3}\frac{1}{D-1}e^x + \frac{1}{-1}e^{0.x}$$

$$= \frac{1}{7}e^{2x} + \frac{2}{3}\cdot\frac{xe^x}{1!} - 1$$

Hence the general solution is $y = c_1e^x + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}x}{2} + c_3\sin\frac{\sqrt{3}x}{2}\right] + \frac{1}{7}e^{2x} + \frac{2}{3}xe^x - 1.$

7) The given differential equation is

$$\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{g}{b}x = \frac{ga}{b}$$

Hence the auxiliary equation is

$$m^2 + g/b = 0$$

$$\therefore m = \pm i\sqrt{g/b}$$

$$\frac{1}{\left(D^2+\frac{g}{b}\right)}\frac{ga}{b} = \frac{ga}{b}\frac{1}{D^2+\frac{g}{b}}e^{0x} = \frac{ga}{b}\frac{1}{0+g/b}e^{0x} = a$$

$$C.F. = c_1\cos(\sqrt{g/b}\,t) + c_2\sin(\sqrt{g/b}\,t)$$

$$\therefore x = c_1\cos(\sqrt{g/b}\,t) + c_2\sin(\sqrt{g/b}\,t) + a$$

$$\therefore \frac{dx}{dt} = \sqrt{g/b}\left[-c_1\sin(\sqrt{g/b}\,t) + c_2\cos(\sqrt{g/b}\,t)\right]$$

$\therefore$ at $t=0$, $x=a'$ gives $a' = c_1 + a \Rightarrow c_1 = (a'-a)$

at $t=0$, $\dfrac{dx}{dt}=0$ gives $c_2 = 0$

$$\therefore x = (a'-a)\cos(\sqrt{g/b}\,t) + a$$

$D^2 - 6D + 8 = 0$

$\Rightarrow D^2 - (4D + 2D) + 8 = 0$

$\Rightarrow (D-4)(D-2) = 0$

$\therefore C.F. = c_1 e^{4x} + c_2 e^{2x}$

$$\frac{1}{(D-2)(D-4)}\left(e^{2x}+1\right)^2$$

$$= \frac{1}{(D-2)(D-4)}\left(e^{4x} + 2e^{2x} + 1\right)$$

$$= \frac{1}{2}\frac{1}{D-4}e^{4x} + 2\cdot\frac{1}{-2}\frac{1}{D-2}e^{2x} + \frac{1}{8}e^{0}$$

$$= \frac{1}{2}\frac{xe^{4x}}{1!} - \frac{xe^{2x}}{1!} + \frac{1}{8}$$

$$\therefore y = c_1 e^{4x} + c_2 e^{2x} + \frac{x}{2}e^{4x} - xe^{2x} + \frac{1}{8}.$$

$f(D) = \phi(D^2) \quad \phi(-a^2) \neq 0$

$$\frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax$$

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax$$

$$\frac{1}{f(D)}\sin ax = \frac{1}{f_1(D^2) + D f_2(D^2)}\sin ax = \frac{1}{f_1(-a^2) + D f_2(-a^2)}\sin ax$$

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$$

(*) $\frac{1}{D^2+a^2}\sin ax$ = imaginary part of $\frac{1}{D^2+a^2}(\cos ax + i\sin ax)$

= imaginary part of $\frac{1}{D^2+a^2}e^{iax} = -\frac{x}{2a}\cos ax$

Now, $\frac{1}{D^2+a^2}e^{iax} = \frac{1}{D^2-(ia)^2}e^{iax}$

$$= \frac{1}{(D+ia)(D-ia)}e^{iax} = \frac{1}{2ia}\frac{1}{D-ia}e^{iax}$$

$$= \frac{1}{2ia}\frac{x}{1!}e^{iax} = \frac{x}{2ia}(\cos ax + i\sin ax) = -i\frac{x}{2a}\cos ax + \frac{x}{2a}\sin$$

$$\frac{1}{D^2+a^2}\cos ax = \text{real part of } \frac{1}{D^2+a^2}(\cos ax + i\sin ax)$$

$$= \text{real part of } \frac{1}{D^2+a^2} e^{iax}$$

Now, $\frac{1}{D^2+a^2} e^{iax} = e^{iax}\frac{1}{(D+ia)^2+a^2}\cdot 1 = e^{iax}\frac{1}{D^2+2iaD}\cdot 1$

$$= e^{iax}\frac{1}{D(D+2ia)}\cdot 1 = \frac{e^{iax}}{2ia}\frac{1}{D\left(1+\frac{D}{2ia}\right)}\cdot 1$$

$$= \frac{e^{iax}}{2ia}\frac{1}{D}\left(1+\frac{D}{2ia}\right)^{-1}\cdot 1 = \frac{e^{iax}}{2ia}\frac{1}{D}\left(1+\frac{D}{2ia}+\cdots\right)1$$

$$= \frac{e^{iax}}{2ia}\frac{1}{D}\cdot 1 = \frac{e^{iax}}{2ia}\int dx = \frac{x e^{iax}}{2ia} = \frac{x(\cos ax + i\sin ax)}{2ia}$$

$$= \frac{x}{2a}(-i\cos ax + \sin ax)$$

$$\therefore \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax = \frac{x}{2a}\int \cos ax\, dx$$

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax = \frac{x}{2}\int \sin ax\, dx$$

(a) $D^2+1=0$

$\Rightarrow D = \pm i$

$\therefore$ C.F. $= c_1\cos x + c_2\sin x$

$$\frac{1}{D^2+1}\cos 2x$$

$$= \frac{1}{-4+1}\cos 2x = -\frac{1}{3}\cos 2x$$

(b) $D^2+9=0$

$\Rightarrow D = \pm 3i$

C.F. $= c_1\cos 3x + c_2\sin 3x$

$$\frac{1}{D^2+9}\cos 4x = \frac{1}{-16+9}\cos 4x = -\frac{1}{7}\cos 4x$$

$\therefore y = c_1$

$D^2 - 3D + 2 = 0$

$\Rightarrow (D-2)(D-1) = 0$

$\therefore D = 1, 2.$

$\therefore C.F. = C_1 e^{x} + C_2 e^{2x}$

$$\frac{1}{D^2 - 3D + 2} \sin 3x$$

$$= \frac{1}{-9 - 3D + 2} \sin 3x$$

$$= \frac{-1}{7 + 3D} \sin 3x$$

$$\cancel{= -\frac{1}{7}\left(1 + \frac{3}{7}D\right)^{-1} \sin 3x}$$

$$= -(3D - 7) \frac{1}{(3D-7)(3D+7)} \sin 3x$$

$$= -(3D - 7) \cdot \frac{1}{9D^2 - 49} \sin 3x$$

$$= -(3D - 7) \cdot \frac{1}{-81 - 49} \sin 3x$$

$$= \frac{1}{130} (3D - 7) \sin 3x$$

$$= \frac{1}{130} \left(\cancel{3} \cdot \frac{\cos 3x}{\cancel{3}} - 7 \sin 3x \right)$$

$$= \frac{1}{130} \left(\cos 3x - 7 \sin 3x \right)$$

$$\therefore y = C_1 e^{x} + C_2 e^{2x} + \frac{1}{130} \left(\cos 3x - 7 \sin 3x \right)$$

(9) $D^3 + a^2 D = 0$

$\Rightarrow D(D^2 + a^2) = 0$

$\therefore D = 0, \pm ia$

$\therefore C.F. = C_1 + C_2 \cos ax + C_3 \sin ax$

$$\frac{1}{D(D^2 + a^2)} \sin ax = \frac{1}{D^2 + a^2} \cdot \left(\frac{1}{D} \sin ax \right)$$

$$= \frac{1}{D^2 + a^2} \left(-\frac{1}{a} \cos ax \right)$$

$$= -\frac{1}{a} \frac{1}{D^2 + a^2} \cos ax$$

$$= -\frac{1}{a} \frac{x}{2} \int \cos ax \, dx = -\frac{x}{2a^2} \sin ax$$

(c) $\frac{d^3x}{dy^3} + b^2 \frac{dx}{dy} = \sin by$

$\Rightarrow (D^3 + b^2 D)x = \sin by$

A.E. $D^3 + b^2 D = 0$

$\Rightarrow D(D^2 + b^2) = 0$

$\therefore D(D+ib)(D-ib) = 0$

$D = 0, \pm ib$

C.F. $= c_1 + c_2 \cos by + c_3 \sin by$

$\therefore y = c_1 + c_2 \cos by + c_3 \sin by - \frac{x}{2b^2} \sin by$

$\frac{1}{D(D^2+b^2)} \sin by = \frac{1}{D^2+b^2} \frac{1}{D} \sin by = \frac{1}{D^2+b^2} \int \sin by \, dy$

$= \frac{1}{D^2+b^2}\left(-\frac{1}{b} \cos by\right) = -\frac{1}{b} \frac{1}{D^2+b^2} \cos by$

$= -\frac{1}{b}\left[\text{Real part of } \frac{1}{D^2+b^2} e^{iby}\right]$

$\cancel{= -\frac{1}{b}\left[\text{Real part of } \frac{1}{-(ib)^2+b^2} e^{iby}\right]}$

$\cancel{= -\frac{1}{b}\left[\text{Real part of } \frac{1}{2b^2}\right.}$

$= -\frac{1}{b}\left[\text{Real part of } e^{iby} \frac{1}{D^2+b^2-b^2+2ibD} \cdot 1\right]$

$= -\frac{1}{b}\left[\text{Real part of } e^{iby} \frac{1}{D(D+2ib)} \cdot 1\right]$

$= -\frac{1}{b}\left[\text{Real part of } \frac{e^{iby}}{2ib} \frac{1}{D} \cdot 1\right]$

$= -\frac{1}{b}\left[\text{Real part of } \frac{x}{2ib}(\cos by + i \sin by)\right]$

$= -\frac{1}{b} \cdot \frac{x}{2b} \sin by = -\frac{x}{2b^2} \sin by$

$$\frac{1}{(D-1)^2(D^2+1)^2}\sin x$$

$$= \frac{1}{(D^2+1)^2}\frac{1}{(D-1)^2}\sin x$$

$$= \frac{1}{(D^2+1)^2}(D+1)^2\frac{1}{(D^2-1)^2}\sin x$$

$$= \frac{1}{(D^2+1)^2}(D+1)^2\frac{1}{(-1-1)^2}\sin x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(D^2+2D+1)\sin x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(-\sin x + 2\cos x + \sin x)$$

$$= \frac{2}{4}\frac{1}{(D^2+1)^2}\cos x$$

$$= \frac{1}{2}\left[\text{Real part of } \frac{1}{(D^2+1)^2}e^{ix}\right]$$

$$= \frac{1}{2}\left[\text{Real part of } e^{ix}\frac{1}{\{(D+i)^2+1\}^2}\cdot 1\right]$$

$$= \frac{1}{2}\left[\text{Real part of } e^{ix}\frac{1}{(D^2+2iD)^2}\cdot 1\right]$$

$$= \frac{1}{2}\left[\text{Real part of } e^{ix}\frac{1}{D^2(D+2i)^2}\cdot 1\right]$$

$$= \frac{1}{2}\left[\text{Real part of } \frac{e^{ix}}{4i^2}\frac{1}{D^2}\frac{1}{\left(1+\frac{D}{2i}\right)^2} e^{0x}\right]$$

$$= \frac{1}{2}\left[\text{Real part of } \frac{e^{ix}}{4i^2}\frac{1}{D^2}\frac{e^{0x}}{(1+0)^2}\right]$$

$$= \frac{1}{2}\left[\text{Real part of } \left(-\frac{e^{ix}}{4}\right)\frac{1}{D^2}\cdot 1\right]$$

$$= \frac{1}{2}\left[\text{Real part of } -\frac{x^2}{8}(\cos x + i\sin x)\right]$$

$$= -\frac{x^2}{16}\cos x$$

6. $D^4 - m^4 = 0 \Rightarrow (D^2+m^2)(D+m)(D-m) = 0$

$\therefore D = m, -m, \pm im$

$$C.F. = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

$$P.I. = \frac{1}{D^4 - m^4}\sin mx = \frac{1}{(D^2+m^2)(D^2-m^2)}\sin mx$$

$$= \frac{1}{-m^2-m^2}\,\frac{1}{D^2+m^2}\sin mx = -\frac{1}{2m^2}\,\frac{1}{D^2+m^2}\sin mx$$

$$= -\frac{1}{2m^2}\left[\text{\cancel{Real} Im part of } \frac{1}{D^2+m^2}e^{imx}\right]$$

$$= -\frac{1}{2m^2}\left[\text{Im. part of } e^{im}\frac{1}{(D+im)^2+m^2}1\right]$$

$$= -\frac{1}{2m^2}\left[\text{Im. part of } e^{im}\frac{1}{D^2+2imD}1\right]$$

$$= -\frac{1}{2m^2}\left[\text{Im. part of } e^{im}\frac{1}{D(D+2im)}e^{0x}\right]$$

$$= -\frac{1}{2m^2}\left[\text{Im part of } \frac{e^{im}}{2im} fdx\right]$$

$$\frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

$y = y_n$ at $x = x_n$. Then $[x_0, x_n]$ divided into n equispaced ~~range~~ points by $x_0, x_1, \ldots, x_n$ given by $x_i = x_0 + ih$ and $h = x_i - x_{i-1}$.

Assuming that $f(x,y) \simeq f(x_{r-1}, y_{r-1})$ in the range & $x \in [x_{r-1}, x_r]$

$$\int_{x_{r-1}}^{x_r} dy = \int_{x_{r-1}}^{x_r} f(x,y)\, dx$$

$$\Rightarrow \quad y_r - y_{r-1} = f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$$

$$\Rightarrow \quad y_r = y_{r-1} + h\, f(x_{r-1}, y_{r-1}). \quad \text{for } r = 1, 2, \ldots$$

$$y_1 = y_0 + h\, f(x_0, y_0), \quad y_2 = y_1 + h\, f(x_1, y_1), \ldots$$

$$y_n = y_{n-1} + h\, f(x_0, y_0).$$

1. The given is $\frac{dy}{dx} = x^3 + y$, where $f(x,y) = x^3 + y$ and $y_0 = 1$ a[illegible]
$x_0 = 0$ and we have get y_n at $x_n = 0.02$ with step length $h =$ [illegible]
By Euler's method

$$\therefore \quad y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.01) \times (0 + 1)$$
$$= 1.0100$$

$$\therefore \quad y_2 = y_1 + h\, f(x_g, y_g)$$
$$= 1.0100 + (0.01) \times \{(0.01)^3 + 1.0100\}$$
$$= 1.0201$$

$$\therefore \quad y(0.02) = 1.0201.$$

0.01
0.01
.0001
0.01
.000001
1.0100
0.000001
1.010001
0.1
.1010001
1.0100
1.1110001

$f(x,y) = x^2+y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.05$

$$y_1 = y(0.05) = y_0 + h\, f(x_0, y_0) = 0 + 0.05 \times (0+0) = 0.0000$$

$$y_2 = 0 + 0.05\left[(0.05)^2 + (0.0000)^2\right] = 0.0001$$

$$y_3 = y_2 + h\, f(x_2, y_2) = 0.0001 + 0.05 \times \left[(0.10)^2 + (0.0001)^2\right]$$

$$= 0.0006.$$

The given differential eqn is $\frac{dy}{dx} = -\frac{y}{1+x}$ with the initial condition $y(0.3) = 2$. Then here, we have $f(x,y) = -\frac{y}{1+x}$ and $x_0 = 0.3$ and $y_0 = 2$ with $h = 0.1$, By Euler's method we have the formula for successive approximations

$$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1}).$$

Then,
$$y_1 = y(0.4) = y_0 + h\, f(x_0, y_0) = 2 + (0.1)\times\left[-\frac{2}{1+0.3}\right]$$

$$= 1.84615$$

$$y_2 = y(0.5) = y_1 + h\, f(x_1, y_1) = 1.84615 + (0.1)\times\left[-\frac{1.84615}{1+0.4}\right]$$

$$= 1.71428$$

Here $x_0 = 0$ and $y_0 = 1$ and $f(x,y) = xy$ with $h = 0.2$. Here we use Euler's method and the approximation formula is given by

$$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1}).$$

$$\therefore y_1 = y(0.2) = 1 + 0.2 \times (0 \times 1) = 1$$

$$y_2 = y(0.4) = 1 + 0.2 \times (0.2 \times 1) = 1.04$$

$$y_3 = y(0.6) = 1.04 + 0.2 \times (0.4 \times 1.04) = 1.1232$$

$$y_4 = y(0.8) = 1.1232 + 0.2 \times 0.6 \times 1.1232 = 1.257984$$

$$1.257984 + 0.2 \times 0.8 \times 1.257984 = 1.45926144$$